Entanglement and Classical Polarization States

Xiao-Feng Qian* and J. H. Eberly
Rochester Theory Center and the Department of Physics & Astronomy
University of Rochester, Rochester, New York 14627

We identify classical light fields as physical examples of non-quantum entanglement. A natural measure of degree of polarization emerges from this identification, and we discuss its systematic application to any optical field, whether beam-like or not.

From Christian Huygens' explanation of the fascinating birefringent property of the crystals called Iceland spar (i.e., calcite), through the much later work of Sir George Stokes [1], the formulation of polarization theory has continuously evolved and is now well established in terms of field correlation functions [2]. The concept of degree of polarization is based on the coherence matrix or polarization matrix constructed from these functions. But, even after centuries of attention to this basic property of optical fields, fascinating new issues concerning polarization have emerged in the past two decades.

The familiar measures of polarization come from the treatment of light as a beam. This implies a given direction of propagation, and thus a specific transverse plane. But development of highly non-paraxial fields, use of very narrow-aperture imaging systems, recognition of associated propagation questions, and probing of fully three-dimensional fields as in hohlraums, all point to necessary modifications of polarimetry beyond the traditional picture [3–5].

Questions of definition and of principle are also in need of answers. For an electromagnetic field without a clear transverse plane or even cylindrical symmetry the relevance of measures such as the conventional degree of polarization must be reconsidered [6], and alternative approaches have been proposed [7–12]. The same is true of experimentally important measures, such as the Mueller matrices [13, 14]. In addition, pioneering measurements [15–18] have highlighted two-photon and multi-photonic views of polarization [19, 20], and polarization measures for nonlinear classical optical waves have been considered [21]. Here we describe a basis for comprehensive consideration of interrelated questions by addressing the issue of dimensionality in a unified way.

Of course, dimensionality is trivially engaged in converting the field from planar-transverse to non-planar, which only requires the natural extension

$$\mathbf{E} = \mathbf{x}E_x + \mathbf{y}E_y \Longrightarrow \mathbf{x}E_x + \mathbf{y}E_y + \mathbf{z}E_z, \quad (1)$$

in order to take into account a third component of the field. However, an entirely comprehensive treatment of polarization questions should begin by first noting that two independent vector spaces are employed in each realization of **E**, and second that the two spaces are entangled [22]. Entanglement as a technical term means just that **E**, in (1), is a tensor product of "lab space" unit vectors such as **x** and **y**, and functions E_x and E_y that are vectors in a statistical "function space" of continuous normed functions (typically taken to depend on space-time or space-frequency).

The light field uses at least one vector from each of these distinct spaces, and in the general case it is not possible to convert (1) to the completely polarized form $\mathbf{E} = \mathbf{u}F$, in which the two spaces appear only in a factored direct product. Such a (typically unlikely) form is called by the equivalent terms "separable" or "factorable" or "non-entangled". That is, determining degree of polarization is the same as determining degree of factorization (separability) of the two spaces, i.e., absence of entanglement. This fact can be exploited.

Both inner and outer products play a role in polarization theory. For convenience we first consider a beam and write the fundamental polarization outer product,

$$|\mathbf{E}\rangle\langle\mathbf{E}| = \Big(|\mathbf{x}\rangle|E_x\rangle + |\mathbf{y}\rangle|E_y\rangle\Big)\Big(\langle\mathbf{x}|\langle E_x| + \langle\mathbf{y}|\langle E_y|\Big), \quad (2)$$

as the field intensity times a normalized hermitian outer product $\mathcal{W}$, i.e., $I\,\mathcal{W} = |\mathbf{E}\rangle\langle\mathbf{E}|$. Here the angle brackets explicitly make the point mentioned above that the unit vectors and the field components are members of different vector spaces. This is a quantum-like notation that will be helpful, but no quantum properties will be introduced.

With the intensity $I = \langle E_x|E_x\rangle + \langle E_y|E_y\rangle$ factored out, we can write

$$|\mathbf{E}\rangle = \sqrt{I}\Big(\cos\theta|\mathbf{x}\rangle|e_x\rangle + \sin\theta|\mathbf{y}\rangle|e_y\rangle\Big), \quad (3)$$

and this allows $\mathcal{W}$ to be written:

$$\mathcal{W} = \Big(\cos\theta|\mathbf{x}\rangle|e_x\rangle + \sin\theta|\mathbf{y}\rangle|e_y\rangle\Big)\Big(\langle\mathbf{x}|\langle e_x|\cos\theta$$
$$+ \langle\mathbf{y}|\langle e_y|\sin\theta\Big), \quad (4)$$

where assignment of the relative amplitudes via sine and cosine factors allows the components $|e_x\rangle$, $|e_y\rangle$ to be unit-normalized: $\langle e_i|e_i\rangle = 1$. We take account of the generally non-zero correlation between the field's components by introducing the magnitude and phase of the cross correlation as

$$\langle e_x|e_y\rangle \equiv \alpha = |\alpha|e^{i\delta}. \quad (5)$$

Although rarely mentioned, each of the two separate vector spaces has its own polarization matrix. These are

---

*Electronic address: xfqian@pas.rochester.edu

reduced-state tensors, i.e., traced over one space independent of the other. The "normal" polarization matrix is obtained by tracing over the function space. We can denote it as $\mathcal{W}_{\text{lab}} = Tr_{\text{fcn}}(\mathcal{W})$, and calculate it by a diagonal sum over any complete set of orthonormal vectors in the function space (but only in the function space), a set that we can label $\{|\phi_m\rangle\}$. For short we will temporarily use p and q to stand for x or y, and then obtain

$$
\begin{aligned}
I\,\mathcal{W}_{\text{lab}} &= I\,Tr_{\text{fcn}}(\mathcal{W}) = \Sigma_m \langle \phi_m | \mathbf{E} \rangle \langle \mathbf{E} | \phi_m \rangle \\
&= \Sigma_{p,q} |\mathbf{p}\rangle \langle \mathbf{q} | \langle E_q | E_p \rangle, \ \ (p, q = x \text{ or } y), \quad (6)
\end{aligned}
$$

which we recognize as a 2×2 tensor in the lab space in the basis defined by $|\mathbf{x}\rangle$ and $|\mathbf{y}\rangle$. When the intensity is factored out we have a familiar matrix expression with the required unit trace:

$$
\mathcal{W}_{\text{lab}} = Tr_{\text{fcn}}(\mathcal{W}) = \begin{bmatrix} \cos^2\theta & \alpha\cos\theta\sin\theta \\ \alpha^*\sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \quad (7)
$$

The less familiar other polarization matrix is obtained by tracing over the lab space. This is trivially done via the projections $|\mathbf{x}\rangle\langle\mathbf{x}|$ and $|\mathbf{y}\rangle\langle\mathbf{y}|$, but the result will not be in standard form because of the non-correlation of E_x and E_y. We can overcome this by rewriting $|\mathbf{E}\rangle$ as a sum of a pair of statistically orthogonal components. If we choose $|e_x\rangle$ as one of them, we will denote its partner by $|\bar{e}_x\rangle$, with $\langle \bar{e}_x | e_x \rangle \equiv 0$. Then $|e_y\rangle$ becomes a combination of both components, $\alpha |e_x\rangle + \beta |\bar{e}_x\rangle$, so

$$
|\mathbf{E}\rangle / \sqrt{I} = (\cos\theta|\mathbf{x}\rangle + \alpha\sin\theta|\mathbf{y}\rangle)|e_x\rangle + \beta\sin\theta|\mathbf{y}\rangle|\bar{e}_x\rangle, \ (8)
$$

where α is defined in (5), and $|\alpha|^2 + |\beta|^2 = 1$. Then the reduced polarization tensor for the function space $\mathcal{W}_{\text{fcn}} = Tr_{\text{lab}}(\mathcal{W})$ has the form:

$$
\mathcal{W}_{\text{fcn}} = \begin{bmatrix} \cos^2\theta + |\alpha|^2\sin^2\theta & \alpha^*\beta\sin^2\theta \\ \alpha\beta^*\sin^2\theta & |\beta|^2\sin^2\theta \end{bmatrix}. \quad (9)
$$

These results serve as background in addressing polarization of non-paraxial or non-beam light fields, i.e., the three dimensional fields expressed as given in (1), and non-trivially entangled. We recall the observation that (1) has the character of an entangled description of two parties jointly, where the "parties" here are the two distinct aspects (or degrees of freedom) of the light field, namely the lab space direction of the optical field and the statistical function space characterizing the strength of the field. This suggests making a Schmidt-type analysis [23, 24], as frequently used in discussions of quantum entanglement data in few-mode and multi-mode photonic contexts [25–29].

The Schmidt theorem applies to any kind of two-party vector, whether quantum or not, and we begin by calculating the same two reduced matrices $\mathcal{W}_{\text{lab}}$ and $\mathcal{W}_{\text{fcn}}$ defined above, except that now they are three-dimensional. Since they arise from a common hermitean $\mathcal{W}$, which is the 3×3 analog of (4), they share the same three (real) eigenvalues, $\kappa_1^2, \kappa_2^2, \kappa_3^2$. Their eigenvectors [30] are

orthonormal and occur in pairs. The Schmidt-form result enables the optical field to be written immediately in a perfectly organized form in terms of these eigenvalues and eigenvectors:

$$
|\mathbf{E}\rangle / \sqrt{I} = \kappa_1 |\mathbf{u}_1\rangle|f_1\rangle + \kappa_2|\mathbf{u}_2\rangle|f_2\rangle + \kappa_3|\mathbf{u}_3\rangle|f_3\rangle. \quad (10)
$$

Here perfect organization means that orthogonality conditions apply in both vector spaces at the same time: $\langle f_i | f_j \rangle = \langle \mathbf{u}_i | \mathbf{u}_j \rangle = \delta_{ij}$, and since intensity has been factored out, the three κs are normalized on the surface of a unit sphere: $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 1$.

This Schmidt decomposition of the field is unique up to a rotation at most, and allows interesting questions to be answered by inspection. For example, it is obvious that (10) can take the completely polarized (fully factored and so non-entangled) form $\mathbf{E} = \mathbf{u}F = \sqrt{I}\,\mathbf{u}\,f$ only when two of the κs have the value zero. Because the three $|f\rangle$ vectors are mutually statistical orthogonal, no other κ values can produce a completely polarized result. For such a field, no matter which lab direction $\mathbf{v}$ is used for a projective measurement of $\mathbf{E}$, the statistical-functional features of that component will always be exactly those of the function f. And similarly, no matter what projection in function space is employed, the projected field's direction will always be $\mathbf{u}$ in lab space.

The connection to standard polarization measures is not complicated. Obviously the conventional definition [2] of degree of polarization P, designed for a beam-type field, is sensible only when there is zero projection along one of the three u's, say along u_3, in which case $\kappa_3 = 0$. Then P has several compact expressions [2, 6, 31]:

$$
P^2 = 1 - 4Det(\mathcal{W}_{\text{lab}}) = |\kappa_1^2 - \kappa_2^2|^2 = 1 - 2(1 - 1/K), \ (11)
$$

where K will be introduced below. As is well known [6], the beam-based definition resists being generalized to the three dimensional case when none of the κ_is is zero in (10). In addition, with $\kappa_3 = 0$, the field cannot point even slightly into the direction $\mathbf{u}_3$. Thus to call it "unpolarized" is not fully sensible even if $\kappa_1 = \kappa_2$, making P zero.

The Schmidt decomposition automatically provides a very useful "weight" parameter K [32], which counts the non-integer effective number of dimensions needed by the optical field. The expression for K is

$$
K = 1/[\kappa_1^4 + \kappa_2^4 + \kappa_3^4], \quad (12)
$$

which gives greater weight to the vector directions with the larger absolute κ values. It's easy to see that K lies between 1 and 3 and incorporates the beam-type field case automatically when any one of the three κ's is zero.

$K = 1$ occurs if $\kappa_2 = \kappa_3 = 0$, which signals a one-term $\mathbf{E}$, i.e., completely polarized light, obtainable only if the original field components could have been rotated into a fully factored form: $\mathbf{E} = \mathbf{u}F$. This is of course generally not possible. The farthest departure occurs for the value $K = 3$, when $\kappa_j^2 = 1/3$ for all three components, and
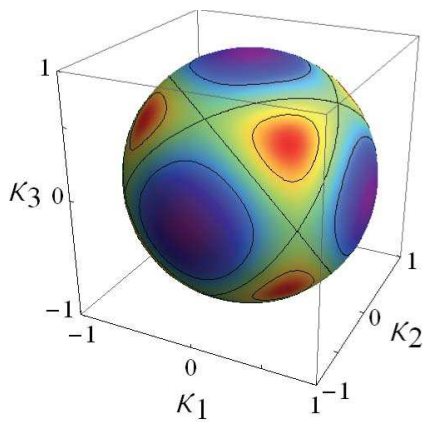
FIG. 1: The entanglement measure K varies from 1 to 3 over the unit polarization sphere. The purple zone-centers touch the surfaces of the cube (K=1, completely polarized), and the red centers are completely unpolarized (K=3). Planar-polarized fields ($K = 2$ or P=0) are located at the corners of the triangular regions where one $\kappa_j^2 = 0$ and the others equal 1/2. The mesh lines locate partially polarized values $K = 3/2, 2, 5/2$.

all f_j are equally intense. This is maximal entanglement and also what is sensibly called a completely unpolarized field. Intermediate values of K represent intermediate degrees of entanglement (partially polarized light). All of these conditions are associated with κ values that identify points on the unit sphere in Fig. 1.

In summary we have reformulated polarization theory as entanglement analysis. The Schmidt theorem approach automatically provides an optimum expression for any light field by identifying its orthogonal directions in lab space, $\mathbf{u}_i$, and its associated amplitudes f_i in statistical function space. This is possible because every optical field is a quantity existing simultaneously in those two independent vector spaces. The interpretation of degree of polarization naturally corresponds to the separability between the two spaces for both planar and non-planar cases. In this new perspective, polarization is a characterization of the correlation between the vector nature and the statistical nature of the light field.

[1] G.G. Stokes, Trans. Cambridge Philos. Soc. **9**, 399 (1852).

[2] Following Zernike's seminal insight that the unobservably rapid variations in optical fields mandated a statistical approach, Wolf developed a comprehensive rigorous formulation of polarization as a branch of statistical coherence theory. See E. Wolf, N. Cim. **13**, 1165 (1959), as well as a modern overview in *Introduction to the Theory of Coherence and Polarization of Light*, E. Wolf (Cambridge Univ. Press, 2007).

[3] E.A. Ash and G. Nicholls, Nature **237**, 510 (1972).

[4] D.W. Pohl, W. Denk, and M. Lanz, Appl. Phys. Lett. 44, 651 (1984).

[5] J.C. Petrucelli, N.J. Moore and M.A. Alonso, Opt. Commun. **283**, 4457 (2010).

[6] C. Brosseau, *Fundamentals of Polarized Light: A statistical Optics Approach* (Wiley, New York, 1998).

[7] J.C. Samson, Geophys. J.R. Astron. Soc. **34**, 403 (1973).

[8] T. Carozzi, R. Karlsson, and J. Bergman, Phys. Rev. E **61**, 2024 (2000).

[9] A.B. Klimov, L.L. Sánchez-Soto, E.C. Yustas, J. Söderholm, and G. Björk, Phys. Rev. A **72**, 033813 (2005).

[10] A. Luis, Opt. Commun. **253**, 10 (2005).

[11] T. Setälä, A. Shevchenko, M. Kaivola, and A.T. Friberg, Phys. Rev. E **66**, 016615 (2002).

[12] J. Ellis, A. Dogariu, S. Ponomarenko, and E. Wolf, Opt. Commun. **248**, 333 (2004).

[13] M. Sanjay Kumar and R. Simon, Opt. Commun. **88**, 464 (1992).

[14] B.N. Simon, S. Simon, F. Gori, M. Santarsiero, R. Borghi, N. Mukunda, and R. Simon, Phys. Rev. Lett. **104**, 023901 (2010).

[15] P.A. Bushev, V.P. Karassiov, A.V. Masalov, and A.A. Putilin, Opt. Spectrosc. **91**, 526 (2001).

[16] T. Tsegaye, J. Söderholm, M. Atatüre, A. Trifonov, G. Björk, A.V. Sergienko, B.E.A. Saleh, and M.C. Teich, Phys. Rev. Lett. **85**, 5013 (2000).

[17] A.B. Klimov, G. Björk, J. Söderholm, L.S. Madsen, M. Lassen, U.L. Andersen, J. Heersink, R. Dong, Ch. Marquardt, G. Leuchs, and L.L. Sánchez-Soto, Phys. Rev. Lett. **105**, 153602 (2010).

[18] T.S. Iskhakov, M.V. Chekhova, G.O. Rytikov, and G. Leuchs, Phys. Rev. Lett. **106**, 113602 (2011).

[19] D.N. Klyshko, Phys. Lett. A **163**, 349 (1992).

[20] D. N. Klyshko, J. Exp. Theor. Phys. **84**, 1065 (1997).

[21] A. Picozzi, Opt. Lett. **29**, 1653 (2004).

[22] X.F. Qian and J.H. Eberly, arXiv: 1009.5622 (2010).

[23] A. Ekert and P.L. Knight, Am. J. Phys. **63**, 415 (1995).

[24] J.H. Eberly, Laser Phys. **16**, 921, (2006).

[25] C.K. Law, I.A. Walmsley and J.H. Eberly, Phys. Rev. Lett. **84**, 5304 (2000).

[26] C.K. Law and J.H. Eberly, Phys. Rev. Lett. **92**, 127903 (2004).

[27] M.N. O'Sullivan-Hale, I.A. Khan, R.W. Boyd, and J.C. Howell, Phys. Rev. Lett. **94**, 220501 (2005).

[28] S.S.R. Oemrawsingh, X. Ma, D. Voigt, A. Aiello, E.R. Eliel, G.W. 't Hooft, and J.P. Woerdman, Phys. Rev. Lett. **95**, 240501 (2005).

[29] M.V. Fedorov, M.A. Efremov, A.E. Kazakov, K.W. Chan, C.K. Law, and J.H. Eberly, Phys. Rev. A **72**, 032110 (2005).

[30] The eigenvalue equations are just 3×3 matrix equations: $\mathcal{W}_{\mathrm{lab}}|\mathbf{u}_j\rangle = \kappa_j^2|\mathbf{u}_j\rangle$ and $\mathcal{W}_{\mathrm{fcn}}|f_j\rangle = \kappa_j^2|f_j\rangle$.

[31] For example, see M.V. Fedorov, P.A. Volkov, and J.M. Mikhailova, Phys. Rev. A , in press (2011).

[32] See R. Grobe, K. Rzążewski and J. H. Eberly, J. Phys. B **27**, L503 (1994).